Note on Latent Roots and Vectors of Segments of the Hilbert Matrix

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As a by-product of work on condition numbers, maximum and minimum latent roots, and the corresponding vectors, have been calculated for the Hilbert segments of orders 4, 6, 8, and 10, to about 17 significant digits.

This supplements the data of Fairthorne and Miller [1] by including minimum latent roots and corresponding vectors, and increasing the accuracy. It also verifies their data (after rounding) except that one error was found (see next paragraph) in their published results.

A Hilbert segment of order N is the matrix $\| 1/(m + n - 1) \|; m, n = 1, \cdots, N$. Let H represent a Hilbert segment, T its inverse, λ a latent root, and V a corresponding vector. Then, e.g., $\lambda_1(H_6)$ is the largest latent root of the segment of order 6; $V_N(T_6)$ is the vector corresponding to the smallest latent root of the inverse of H_6. The error in Fairthorne and Miller's article occurs in the third element of $V_1(H_6)$.

The power method [2] was used, with a double-precision floating-point routine on the IBM 650. Smallest latent roots were obtained as $\lambda_N(H_N) = 1/\lambda_1(T_N)$, and verified (to as many significant figures as the method allows) by direct calculation of $\lambda_N(H_N)$ by the power method, $\lambda_N(H)$ being obtained as $\lambda_1(H - pI)$, where I is the identity matrix and p is slightly greater than $\lambda_1(H)$. The Nth vectors are given as $V_N(H_N) = V_1(T_N)$, because the method gives greater accuracy here for V_1 than for V_N and because T has no input error.

Terminal digits are uncertain by not more than one, as indicated by convergence rates.

$\lambda_1(H_4)$	$\lambda_N(H_4)$
1. 50021 42800 59242 81	$10^{-4} \times$ 0. 96702 30402 25868 861
$V_1(H_4)$	$V_N(H_4)$
1.	0. 03688 76826 14141 047
0. 57017 20836 63235 83	−0. 41534 92877 80311 17
0. 40677 89880 27529 24	1
0. 31814 09688 73793 96	−0. 65017 12197 33679 82
$\lambda_1(H_6)$	$\lambda_N(H_6)$
1. 61889 98589 24339 1	$10^{-7} \times$ 1. 08279 94845 65549 8
$V_1(H_6)$	$V_N(H_6)$
1	0. 00180 94825 41440 515
0. 58862 85434 25543 2	−0. 05161 82535 94248 58
0. 42832 72844 28956 1	0. 34890 77525 35503 9
0. 33966 18918 38709 5	−0. 90671 76845 78412 7
0. 28252 35879 42149 2	1
0. 24233 78111 22849 5	−0. 39374 11114 93702 0
$\lambda_1(H_8)$	$\lambda_N(H_8)$
1. 69593 89969 21949 4	$10^{-10} \times$ 1. 11153 89663 72442 4

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$V_1(H_8)$	$V_N(H_8)$
1.	−0. 00006 86103 92145 12812
0. 60050 42457 57953 8	0. 00368 78770 51827 661
0. 44267 15540 11918 6	−0. 04826 72545 24498 43
0. 35437 04469 99697 8	0. 26171 33996 76104 1
0. 29691 85784 44507 1	−0. 70574 73471 79618 8
0. 25618 09294 86980 5	1.
0. 22562 93688 08227 6	−0. 71250 91381 80124 8
0. 20179 01870 37918 3	0. 20124 18343 83776 4
$\lambda_1(H_{10})$	$\lambda_N(H_{10})$
1. 75191 96702 65177 5	10^{-13} × 1. 09315 38198 57659 9
$V_1(H_{10})$	$V_N(H_{10})$
1.	0. 00000 27147 13133 60409 7
0. 60899 19143 69650 3	−0. 00023 60612 62959 0383
0. 45313 82989 59421 5	0. 00505 28973 86716 890
0. 36528 60134 02151 0	−0. 04611 60400 49989 25
0. 30775 30474 45501 6	0. 22066 15177 28910 4
0. 26672 51842 93050 8	−0. 60817 67839 54336 8
0. 23580 13079 82484 3	1.
0. 21156 39639 51540 1	−0. 96815 88795 12219 1
0. 19200 51281 86119 1	0. 50907 35851 67138 3
0. 17586 00343 93102 9	−0. 11210 49402 14747 4

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1. R. A. Fairthorne & J. C. P. Miller, "Hilbert's double series theorem and principal latent roots of the resulting matrix," *MTAC*, v. 3, 1949, p. 399.
2. Marvin Marcus, "Basic theorems in matrix theory," *Nat. Bur. Standards, Appl. Math. Ser.* No. 57, U. S. Government Printing Office, Washington, D. C., 1960.